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GENERAL SOLUTION TO THE GASCA-MAEZTU HYPOTHESIS

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Abstract:

To date, the Gasca - Maeztu hypothesis has been proven to be n≤4 degrees.
The article provides proof of hypothesis for any n points and uses a fundamentally new approach to proof. The proof is based on the following two propositions: first, that any line passing through two points coincides, and second, that at least two lines pass through any point in the given situation. Andthe new approach is that the number of binary combinations of the mentioned n points is limited.
That is, after removing a point, when the remaining points are covered with straight lines, matching lines appear because the binary combinations of points are already repeated. Consequently, there willalready be (n + 1) points on some straight lines.

Keywords: *hypothesis, point, line, binary combinations, line coincidence, missing point.*

The statement of the problem is as follows. We have N= $\frac{(n+1)(n+2)2}{}$ points. Prove that if N-1 of these points are covered by n lines (lines should not cross a missing point), then n + 1 points will be found on at least one of the lines.

The solution to the problem is based on the following well known statements.

I. if, for example, the points a, b, c, d are on one line and the points e, f, c, d, g are on another line, then those lines will coincide because both pass through the points c և d : And there will already be 7 points on that line.

II. In case of a problem with this position, at least two lines must pass through the same point.

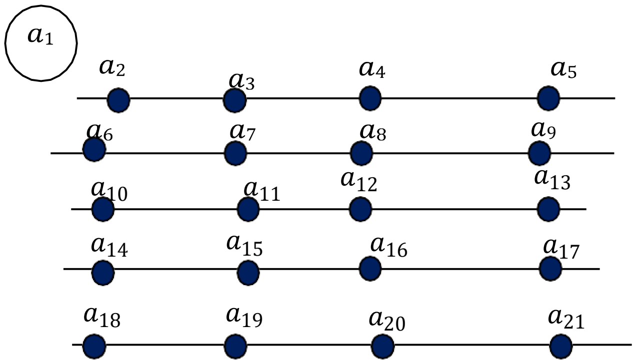
First, let's prove the problem with concrete examples.

1) For example, consider the case n = 5. In this case, the number of points to be covered willbe N = 21 points. Let's denote a_1 , a_2 , a_3 , ..., a_{21}. We need to cover 20 of them with 5 lines so that no line passes through the missing point. And so on, excluding the remaining points.

The solution of the problems based on the provisions of the above I and II is as follows.

It is possible to cover 20 points with five lines in different ways. We consider in advance the case where there are 4 points on each of the lines. We choose this because in version 4,4,4,4,4 the binary combinations of points will be the least.

Indeed, this version gives the minimum number of connected binary pairs $5C_4^2$ =30. And when covering any other option, the sum of the binary combinations of the points on them is greater (for example 5,4,4,4,3 = 31; 5,5,5,3,2 = 34 և etc.). The binary combinations of all 21 elements are equal to C_{21}^2=210. First, by removing, say, a_1 we first cover the 20 points in the following order.



(1)

In (1), the binary combinations of points in each line are equal to 6 (for example, the first line will be $a_2\ a_3$, $a_2\ a_4$, $a_2\ a_5$, $a_3\ a_4$, $a_3\ a_5$, $a_4\ a_5$).

After removing the next points, consider two options for covering the remaining 20 points with 5 straight lines.

a) when from C_{21}^4=11970 possible quadratic combinations we can choose five groups (5 lines with 4 dots on each), in which C_{21}^2=210 one of the possible binary combinations is not repeated.

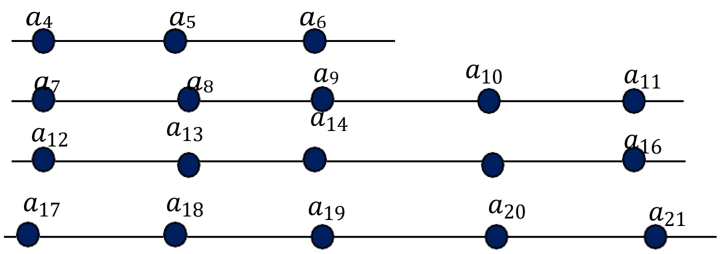
b) we use the straight lines that were obtained for the previous points.

According to option a), if there is no quadrant of points where there is no repetition of the pairs of points in the previous quatrains (straight lines), then according to I, the proof of the problem ends because there will be 6 points on any line.

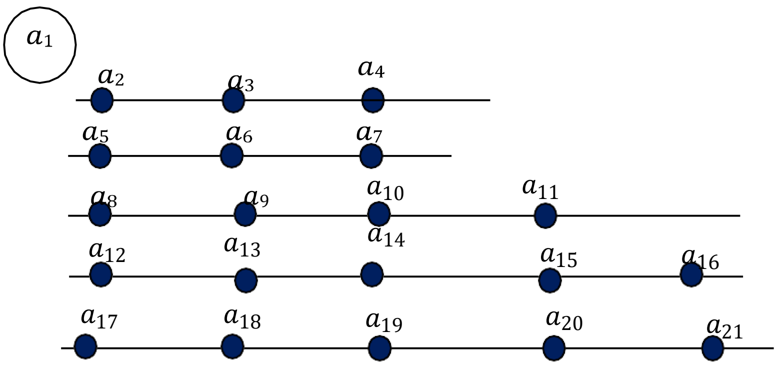
However, if we consider that they are, then since there are 30 different binary combinations in each of the five quartets, so successively removing the points a_2 , a_3 , ... a_6 and a_7 we consume C_{21}^{2}=210 all unique binary combinations. Then, after removing the next point, any of the selected quadruple elements will already have binary combinations that already exist in the previous quadrant (on the line). Therefore, the line representing this quartet will coincide with the line with the corresponding binary combination, and since there are four points on each, then according to I, there will already be 6 points on the line.

What was required to be proved.

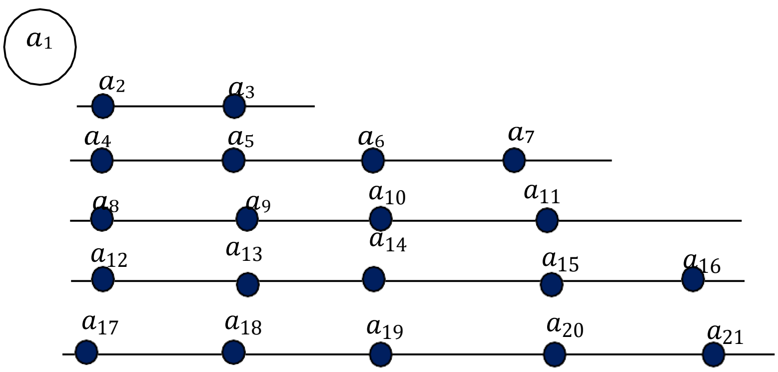
But there are other ways to arrange 20 points on five lines.



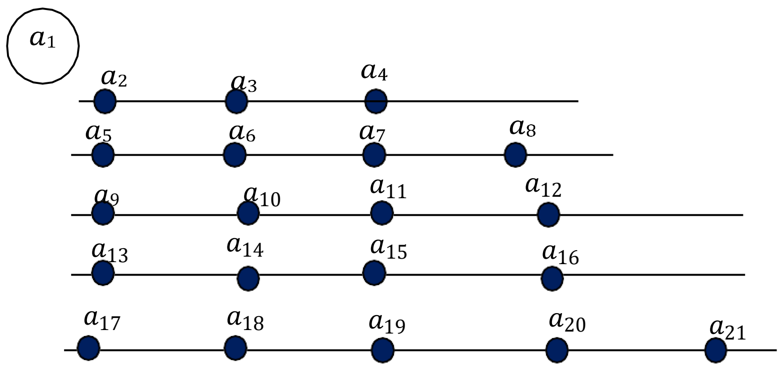
(2)



(3)



(4)



(5)

These are (2), (3), (4) և (5). With the same equation we can prove that in the case of these variants, when covering these points directly, we can make choices C_{21}^2 =210 , C_{21}^3 =1330 , C_{21}^4 =11970 and C_{21}^5 =20349 and with two, three, four and five points. combinations and make new lines. However, after a few points to be removed, the binary combinations C_{21}^2 =210 will be exhausted . Therefore, after the removal of the following points, in any choice of straight points, they will repeat the binary combination that already exists on the previous lines. And finally, on some straight line there will be 6 points.

This way we can prove any n. For example, for n = 7, it is proved that points a_1 , a_2 a_3 , ... a_9 after subtracting a_{10}, the possible binary combinations of all 45 points are exhausted 8 8 points will already be found on any new line.

b) Now consider the option where after the next point removed, some lines (lines) of the version received for the previous point are taken.

For all of the (1), (2), (3), (4) և (5) options considered, starting with the omission of point 3, any point will already be on two different lines. Therefore, when deleting a point, the points on those lines should be arranged on different lines, as new lines should not pass through the removed point.

It is obvious that in options (2), (3) and (4), when we remove a point from the last two lines and use another one from those lines (lines), then the remaining 4 points because we have to put 4 different lines so it immediately turns out that there is already 6 points on it. That's the solution. And if we keep the first two lines (straight lines), then in this case we have to make new rearrangements of the points, which a) does not differ significantly from the version.

In the case of option (5) we get lines with two five-point points, when one of the points of the last line is removed, the other 4 points are redistributed. The discussed options (2), (3) և (4) are obtained again.

So it remains to consider option (1).

Since at least two lines must pass through each point, when removing the next point, the 6 (3 and 3) related points in the next version must be placed on at least 3 different lines. We can not place them on the same line, as they are already connected to the previous line, we will get a line with 6 points at once.

That is, we can take two lines at best from the previous version. Then in the new version we get three quadrants (lines) with new rearrangements of the elements, in which the number of binary combinations will be equal to 3x6 = 18.

In that case, after removing the 10th point, the number of connected binary combinations will be 30 + 180 = 210.

Subsequently, after removing point 10, all non-repeating binary combinations will be exhausted. So after removing the next point, any quadrilateral will already have binary combinations that already existed in the prvious quartet. Then the line representing that quartet will coincide with the given line, there will already be 6 points on that line.

What had to be proved.

And now let's give a general assessment of the solution to the problem.

In the case of n, the total number of points is equal $N=\frac{(n+1)(n+2)}{2}$.

Number of points to be covered:

$$N_c = \frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2} \qquad (6)$$

Since they must be covered by n lines, each covering will fall on the line points

$$n_c = \frac{n(n+3)}{2n} = \frac{n}{2} + \frac{3}{2} \qquad (7)$$

Let us make a remark here. Of course, when covering given points with lines, it is not necessary that all points have equal points. However, since the solution of the problem is based on the number of binary combinations of points, it is worth noting that the sum of binary combinations of the two parts divided by the two parts of the given 2m points is less when there are equal numbers of elements in the group. Really, for example, take the groups m and m lı m + 2 and the group m-2. There will be binary combinations in groups.

$2C^2_m = \frac{2m(m-1)}{2} = m(m-1)$ and $C^2_{m+2} + C^2_{m-2} = \frac{(m+2)(m+1)}{2} + \frac{(m-2)(m-3)}{2} = m(m-1) + 4.$

So the proof is more solid when we take equal points on the lines.

There will be a number of binary combinations for all points.

$$C^2_{\frac{(n+1)(n+2)}{2}} = \frac{\frac{(n+1)(n+2)}{2}\left(\frac{(n+1)(n+2)}{2}-1\right)}{2} \quad (8)$$

Since there are $\frac{n}{2}+\frac{3}{2}$ points on each line in advance, the number of binary combinations associated with each line will be

$$C^2_{\frac{(n+3)}{2}} = \frac{\frac{(n+3)}{2}\left(\frac{n+3}{2}-1\right)}{2} = \frac{(n+1)(n+3)}{8} \quad (9)$$

The analysis shows that when we remove any point and insert the next point, we can take the most $\frac{n}{2}-\frac{1}{2}$ pieces from the previously obtained lines. Since the number of rows (lines) is equal to n, then the number of rows by grouping the points will be equal to n - $(\frac{n}{2}-\frac{1}{2})$ = $\frac{n}{2}+\frac{1}{2}$. In that case, the two new combinations of straight points constructed for a given point of departure will be equal $\frac{(\frac{n}{2}+\frac{1}{2})(n+1)(n+3)}{8}$.

Now determine the next point to be removed N_{lm}, when the sum of the binary combinations of points on the lines covering $\frac{n}{2}+\frac{1}{2}$ exceeds all possible binary combinations, so that at least n + 1 points are already on the line.

We will have

$$\frac{(\frac{n}{2}+\frac{1}{2})(n+1)(n+3)}{8} N_{lm} > \frac{\frac{(n+1)(n+2)}{2}\left(\frac{(n+1)(n+2)}{2}-1\right)}{2}$$

or

$$(\frac{n}{2}+\frac{1}{2})(n+3)N_{lm} > (n+2)(n^2+3n) \quad (10)$$

For example, in case n = 5 it is obtained

$N_{lm} > \frac{280}{24} > 10$ which ꙇ we got during the direct calculation.

In case n = 7 it is obtained

4.10. $N_{lm} > 9.70$ and $N_{lm} > \frac{630}{40} > 15.$

For example, in the case of n = 25 points, we get that from N = 351 points, after removing point 52, covering 350 points with 25 lines, there will already be 26 points (n + 1 points) on at least one of them.

Conclusion

Thus, the work proved the Gasca-Maeztu hypothesis, which had been unsolvable for 40 years. The proof of the hypothesis is based on two simple statements. Andthe it should be noted that the Gasca-Maeztu hypothesis is proved on the basis of the finiteness of binary combinations of points. The work has theoretical and practical significance.

Statements and Declarations

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

The authors declare that they have no conflict of interest.

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