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RANKS, SUBDEGREES AND SUBORBITAL GRAPHS OF SYMMETRIC GROUP S_n ACTING ON ORDERED PAIRS

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Abstract: -

In this research paper, we study the ranks and subdegrees of the symmetric group S_n (n = 3, 4, 5) acting on ordered pairs from the set X = {1, 2 , … , n}. When S_n ($n \geq 4$) acts on ordered pairs from X, the rank is 7. Therefore the main study will be on the ranks and subdegrees of the suborbitals.

The suborbital graphs corresponding to the suborbitals of these actions are also constructed. The graph theoretic properties of these suborbital graphs are also discussed. When S_n ($n \geq 4$) acts on ordered pairs, the suborbital graphs, Γ_1,Γ_2,Γ_5, and Γ_6 corresponding to the non-trivial suborbits, Δ_1 ,Δ_2 , $\Delta_5$and Δ_6 are disconnected, regular and undirected. The suborbital graphs $\Gamma_3$and Γ_4 are disconnected, and directed.

Key words: - *Ranks, Subdegrees, Suborbitals, Suborbital graphs and Ordered pairs.*

INTRODUCTION

In this paper we investigate some properties of the symmetric group S_n (n = 3, 4, 5) acting on ordered pairs from X = {1, 2, … , n}. We also find suborbits and suborbitals of S_n (n = 3, 4, 5) and construct suborbital graphs corresponding to these suborbitals. We shall also discuss some of the graph theoretic properties of these suborbital graphs.

This paper is divided into three parts; with our main results in part two.

In part one, we give definitions and preliminary results needed throughout the paper.

In part two, we investigate some properties of the action of S_n (n = 3, 4, 5) on ordered pairs. We also find the ranks, suborbits and construct suborbital graphs corresponding to the suborbitals of S_n (n = 3, 4, 5). We also discuss the graph theoretic properties of these suborbital graphs. Finally in part three, we give conclusions.

DEFINITIONS AND PRELIMINARIES

We establish background information and results that will be used throughout this paper. It is

1.1 Notations

$\sum_i$ − Sum over i.

$(_b{}^a)$ - a combination b.

S_n - Symmetric group of degree n and order n!.

|G| - The order of a group G.

|G: H| - Index of H in G.

$X^{[2]}$ - The set of ordered pairs from the set X = {1,2, … , n}.

(t,q)- Ordered pair.

X×Y – Cartesian product of X and Y.

1.2 Permutation groups

Definition 1.2.1

Let X be a non-empty set. A permutation of X is a one-to-one mapping of X onto itself.

Definition 1.2.2

Let X be the set {1,2,… , n} , then the symmetric group of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted as $S_n$and has an order n!.

Definition 1.2.3

A permutation of a finite set is even or odd according as it can be expressed as the product of an even or odd number of 2-cycles (transpositions).

1.3 Group actions Definition 1.3.1

Let X be a non-empty set. The group G acts on the left on X if for each g ∈ G and each x ∈ X there corresponds a unique element gx ∈ X such that :

(i) $(g_1g_2)\, x = g_1(g_2x)$, ∀ g_1, g_2 ∈ G and x ∈ X.

(ii) For any x ∈ X, 1x = x, where 1 is the identity in G. The action of G from the right on X can be defined in a similar way. Infact it is merely a matter of taste whether one writes the group element on the left or on the right.

Definition 1.3.3

Let G act on a set X. Then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each x ∈ X, the orbit containing x is called the orbit of x and is denoted by $Orb_G(x)$.

Definition 1.3.4

Let G act on a set X and let x ∈ X. The stabilizer of x in G, denoted by $Stab_G(x)$ is given by $Stab_G(x)$ = {g ∈ G|gx = x}.
Note: $Stab_G(x)$ forms a subgroup of G which is called the isotropy group of x. This subgroup is denoted by G_x.

Definition 1.3.5

If a finite group G acts on a set X with n elements, each g ∈ G corresponds to a permutation σ of X, which can be written uniquely as a product of disjoint cycles. If σ has $\propto_1$ cycles of length 1, $\propto_2$ cycles of length 2, $\propto_3$ cycles of length 3,…, $\propto_n$ cycles of length n; then we say that σ and hence g has a cycle type $(\propto_1,\propto_2, \propto_3, \dots , \propto_n)$.

Definition 1.3.6

Let G act on a set X. The set of elements of X fixed by g∈ G is called the fixed point set of g and is denoted by Fix(g). Thus Fix(g) = {x ∈ X|gx = x}.

Definition 1.3.7

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points x, y ∈ X, there exists g ∈ G such that gx = y.

Theorem 1.3.8 [Krishnamurthy,1985, p. 68].

Two permutations in S_n are conjugate if and only if, they have the same cycle type, and if $g \in S_n$ has a cycle type $(\propto_1, \propto_2, \ldots \propto_n)$, then the number of permutations in S_n conjugate to g is

$$\frac{n!}{\prod_{i=1}^{n} \propto_i !\, i^{\propto_i}}.$$

Theorem 1.3.9 [Orbit –Stabilizer theorem – Rose,1978, p. 72]
Let G act on a set X and let $x \in X$, the $|Orb_G(x)| = |G: Stab_G(x)|$.

Theorem 1.3.10 [Cauchy − Frobenius Lemma − Rotman, 1973, p. 45.]
Let G be a group acting on a finite set X. then the number of G-orbits in X is

$$\frac{1}{|G|}\sum_{g\in G}|Fix\ (g)|.$$

The immediate theorem above is usually but erroneously attributed to Burnside (1911) cf. Neumann (1977).

1.4 Graphs and digraphs Definition 1.4.1
A graph is a diagram consisting of a set V whose elements are called vertices, nodes or points and a set E of unordered pairs of vertices called edges or lines. We denote such a graph by G(V,E) or simply by G if there is no ambiguity of V and E.

Definition 1.4.2
Two vertices u and v are said to be adjacent if there is an edge joining them. This is denoted by {u, v}and sometimes by uv. In this case u and v are said to be incident to such an edge.

Definition 1.4.3
A walk in a graph consists of a finite sequence of edges of the form $v_0, v_1, v_1, v_2, \ldots, v_{m-1}, v_m$. The number m of edges in the walk above is called the length of the walk. A walk is said to be closed if $v_0 = v_m$. A trail is a walk in which all edges are distinct. A path is a walk in which all vertices are distinct. A cycle (circuit) is a closed path. A cycle of length k is called a k-cycle.

Definition 1.4.4
A graph G(V,E) is said to be connected if there is a path between any two of its vertices.

Definition 1.4.5
The girth of a graph G(V,E) is the length of the shortest cycle if any in G(V,E).

Definition 1.4.6
A graph in which every vertex has the same degree is called a regular graph.

Definition 1.4.7
A digraph or a directed graph consists of a finite non-empty vertex set V(G) together with a prescribed collection X of ordered pairs of distinct vertices. The elements of X are directed lines or arcs.

1.5 Suborbits and suborbital graphs Definition 1.5.1
Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. the orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \ldots, \triangle_{r-1}$ of G_x on X are called the suborbits of G. The rank of G is r and the sizes $n_i = |\triangle_i|$ (i= 0,1,2,…,r-1) often called the lengths of the suborbits, are known as subdegrees of G.
It is worth while noting that both r and the cardinalities of the suborbits $\triangle_i$(i= 0,1,…, r-1) are independent of the choice of $x \in X$.

Definition 1.5.2
Let $\triangle$ be an orbit of G_x on X. define $\triangle^* = \{gx | g \in G, x \in g\triangle\}$, then $\triangle^*$ is also an orbit of G_x and is called the G_x – orbit (or the G-suborbit) paired with $\triangle$.
Clearly $|\triangle| = |\triangle^*|$. If $\triangle^* = \triangle$, then $\triangle$ is called a self-paired orbit of G_x.

Theorem 1.5.3 [Wielandt,1964,Section 16.5]
G_x has an orbit different from {x} and paired with itself if and only if G has even order. Observe that G acts on X×X by $g(x,y) = (gx,gy)$, $g \in G$, $x,y \in X$.
If $O \subseteq X\times X$ is a G-orbit, then for a fixed $x \in X$, $\triangle = \{y \in X | (x,y) \in O\}$ is a G_x- orbit.
Conversely, if $\triangle \subseteq X$ is a G_x- orbit, then $O = \{gx, gy) | g \in G, y \in \triangle\}$ is a G-orbit on X×X. We say $\triangle$ corresponds to O.

Theorem 1.5.5. [Sims,1967]
Let G be transitive on X. Then G is primitive if and only if each suborbital graph Γ_i (I = 1, 2, …, r-1), is connected.

2.0 ACTION OF THE SYMMETRIC GROUP S_n ON ORDERED PAIRS

In this part, we investigate the action of G = S_n on $X^{[2]}$, the set of ordered pairs from X = {1,2, … , n}. We also construct and discuss the suborbital graphs associated with this action.
Now, G acts on the set $X^{[2]}$, of all ordered pairs from X by the rule;

$$g(x,y) = (gx,gy)\ ,\ \forall g \in G \text{ and } (x,y) \in X^{[2]}.$$

This part is divided into two Sections. Section 2.1 deals with some general results of permutation groups acting on $X^{[2]}$. Section 2.2 deals with the suborbits of S_n (n = 3, 4, 5) acting on $X^{[2]}$ and the corresponding suborbital graphs.

2.1 SOME GENERAL RESULTS OF PERMUTATION GROUPS ACTING ON $X^{[2]}$

The following two Theorems whose proofs are given will be very useful in the entire part, for the calculations of the number of ordered pairs fixed by g; that is
|Fix (g)| and the number of permutations in G fixing (a, b) and having the same cycle type as g∈ G respectively.

Theorem 2.1.1

Let G be a symmetric group S_n acting on the set X={1,2, … , n} and g∈ G have cycle type $(\propto_1, \propto_2, \dots, \propto_n)$
Then |Fix (g)| in $X^{[2]}$ is given by

$$\frac{\propto_1!}{(\propto_1-2)!}.$$

Proof

For g∈ G and having cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ to fix an ordered pair (a,b) then both a and b must come from cycles of length 1.
In this case the number of ordered pairs fixed by g is

$$2\binom{\propto_1}{2} = 2\left(\frac{\propto_1!}{2!\,(\propto_1-2)!}\right) = \frac{\propto_1!}{(\propto_1-2)!}.$$ □

Theorem 2.1.2

Let G be a symmetric group S_n acting on the set X = {1,2,3, … , n} and let g∈ G have cycle type $(\propto_1, \propto_2, \dots, \propto_n)$.
Then the number of permutations in G fixing (a, b) and having the same cycle type as g is given by

$$\frac{(n-2)!}{1^{(\propto_1-2)}(\propto_1-2)!\prod_{i=2}^{n}\propto_i!\ i^{\propto_i}}.$$

Proof

For a permutation in G having cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ to fix (a,b) then a and b must come from single cycles. The number of permutations in S_n of cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ fixing (1,2) is the same as the number of permutations in S_{n-2} of cycle type $(\propto_1 - 2, \propto_2, \dots, \propto_n)$. Now by Theorem 1.3.10, this number is

$$\frac{(n-2)!}{1^{(\propto_1-2)}(\propto_1-2)!\prod_{i=2}^{n}\propto_i!\ i^{\propto_i}}$$ □

2.2 SUBORBITS OF S_n (3, 4, 5) ACTING ON $X^{[2]}$ AND THE CORRESPONDING SUBORBITAL GRAPHS

2.2.1 SUBORBITS OF G = S_3 ACTING ON $X^{[2]}$ AND THE CORRESPONDING SUBORBITAL GRAPHS Lemma 2.2.1.1

G acts transitively on $X^{[2]}$.
Proof
Let g∈ G have cycle type $(\propto_1, \propto_2, \propto_3)$, then the number of permutations in G having the same cycle type as g is given by Theorem 1.3.10. The number of elements in $X^{[2]}$ fixed by g is given by Theorem 2.1.1.
We now have the following table;

Table 1: Permutations in G and the number of points fixed by g ∈ G

Permutation g ∈ G	No. of permutations	$\|Fix(g)\|$ in $X^{[2]}$	Cycle type $(\alpha_1, \alpha_2, \alpha_3)$
I	1	6	(3,0,0)
(ab)	3	0	(1,1,0)
(abc)	2	0	(0,0,1)
Total	6		

Now, applying Cauchy – Frobenius Lemma we have the number of orbits of G acting on $X^{[2]}$

$$= \frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$

$$= \frac{1}{6}\{(1x6)+(3x0)+(2x0)\} = \frac{1}{6} \text{ x } 6 = 1$$

Therefore G acts transitively on $X^{[2]}$ □

Alternatively we can use the Orbit –Stabilizer Theorem (Theorem 1.3.9) to prove the lemma above. In this case we need to show that the length of the orbit of a point say (1, 2) ∈ $X^{[2]}$ is 6, the same as the number of points in $X^{[2]}$. Meaning that the action of G on $X^{[2]}$ has only one orbit.

Let g ∈ G have cycle type $(\alpha_1, \alpha_2, \alpha_3)$. Then the number of permutations in G fixing (1, 2) and having the same cycle type as g is given by Theorem 2.1.2.

We now have the following table;

Table 2: Number of permutations in G fixing (1,2)

Permutation type	No. Fixing (1,2)	Cycle type $(\alpha_1, \alpha_2, \alpha_3)$
I	1	(3,0,0)
(ab)	0	(1,1,0)
(abc)	0	(0,0,1)
Total	1	

Therefore $|stab_G(1,2)| = 1$

Now, applying the Orbit – Stabilizer Theorem, we get

$$|Orb_G(1,2)| = |G : Stab_G(1,2)|$$

$$= \frac{|G|}{|Stab_G(1,2)|} = \frac{6}{1} = 6.$$

Thus the orbit of (1, 2) is the whole of $X^{[2]}$ and therefore G acts transitively on $X^{[2]}$. □

Lemma 2.2.1.2

The number of orbits of $G_{(1,2)}$ acting on $X^{[2]}$ is 6.

Proof

The Cauchy- Frobenius Lemma helps in counting the number of orbits.

The second and the third columns of the following table can be got by applying Theorems 2.1.2 and 2.1.1 respectively.

Table 3: Permutations in $G_{(1,2)}$ and the number of fixed points

Permutation g in $G_{(1,2)}$	No. of permutations	$\|Fix(g)\|$in $X^{[2]}$
1	1	6

By Cauchy –Frobenius Lemma, we have

$|Orb_{G(1,2)}(1,2)| = 6$

Thus the rank of G acting on $X^{[2]}$ is 6. □These six orbits are;

$Orb_{G(1,2)}(1,2) = \{(1,2)\} = \Delta_o$, the trivial orbit.

$Orb_{G(1,2)}(2,1) = \{(2,1)\} = \Delta_1$, the transpose of the trivial orbit.

$Orb_{G(1,2)}(1,3) = \{(1,3)\} = \Delta_2$, the set of all ordered pairs containing exactly one 1 and of the form (1,a) where a ≠ 2.

$Orb_{G(1,2)}(3,1) = \{(3,1)\} = \Delta_3$, the set of all ordered pairs containing exactly one I and of the form (a,1), a ≠ 2, the transpose of (1,a).

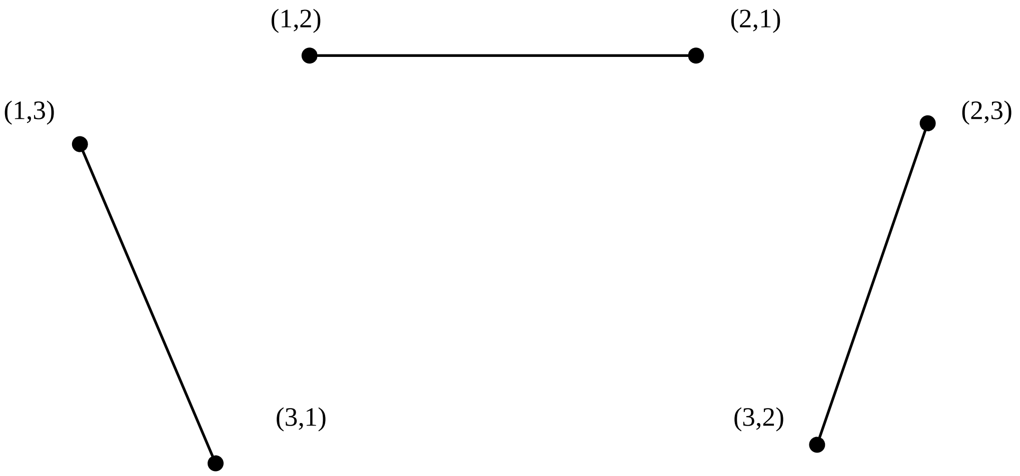
$Orb_{G(1,2)}(2,3) = \{(2,3)\} = \Delta_4$, the set of all ordered pairs containing exactly one 2 and of the form (2,b), where b≠1.

$Orb_{G(1,2)}(3,2) = \{(3,2)\} = \Delta_5$, the set of all ordered pairs containing exactly one 2 and of the form (b,2) , b≠1, the transpose of (2,b).

Thus, the subdegrees of G on $X^{[2]}$ are 1,1,1,1,1,1.

We now discuss the suborbital graphs corresponding to the suborbits determined above.
The suborbital graph corresponding to Δ_o is the null graph. We now consider the remaining suborbits Δ_1, Δ_2, Δ_3,Δ_4 and Δ_5 as follows;
Let V and W be any two distinct ordered pairs from X = {1,2,3}; then
(a) The suborbital $O_1 = \{(g(1,2),g(2,1))|g \in G\}$ (see Section 1.5).
The suborbital graph Γ_1 corresponding to the suborbital O_1 has a directed edge from the ordered pair V to the ordered pair W if and only if the first co-ordinate of V is identical to the second co-ordinate of W and the second co-ordinate of V is identical to the first co-ordinate of W.

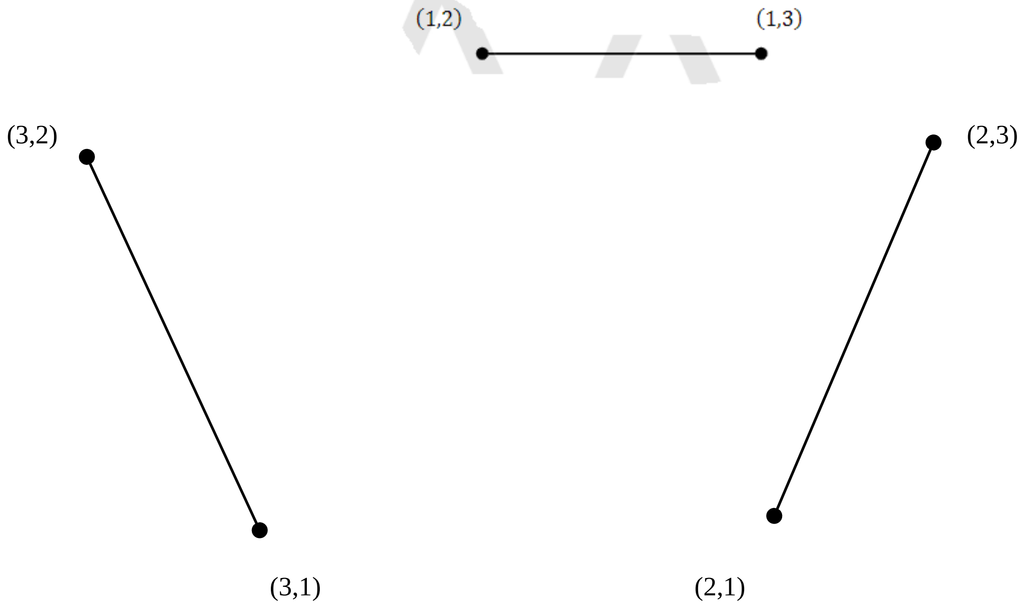
Figure 1: The suborbital graph Γ_1 corresponding to the suborbital O_1



Γ_1 is disconnected , regular of degree 1. Γ_1 is also undirected since its corresponding suborbit is self-paired.

(b) The suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g(1,2), g(1,3)|g \in G\}$.
The suborbital graph Γ_2 corresponding to the suborbital O_2 has a directed edge from the ordered pair V to the ordered pair W if and only if the first co-ordinate of V is identical to the first co-ordinate of W and the second co-ordinate of V is not identical to the second co-ordinate of W.

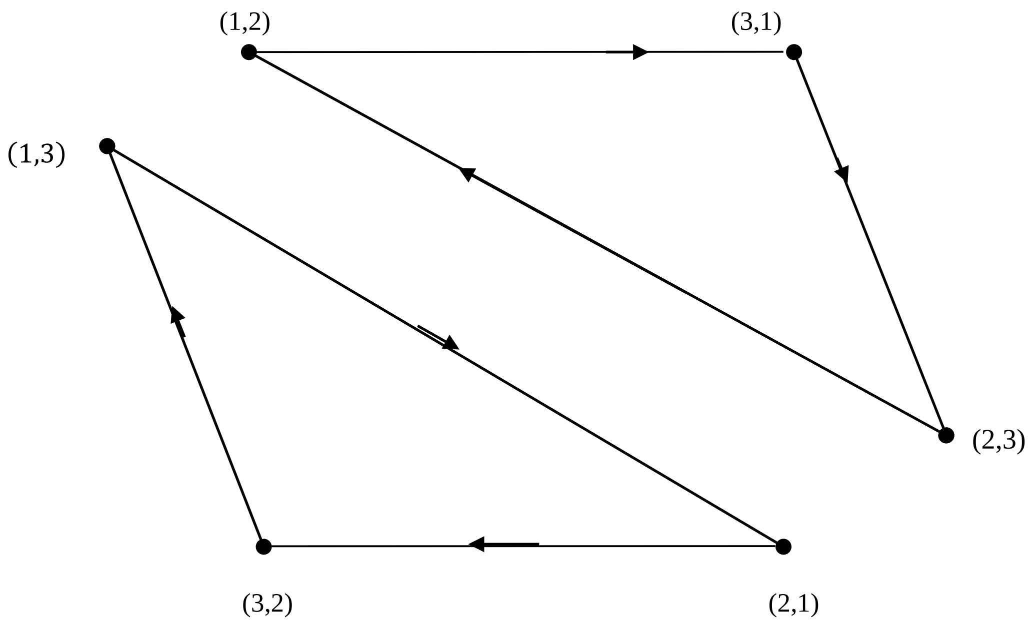
Figure 2: The suborbital graph Γ_2 corresponding to the suborbital O_2



Γ_2 is disconnected , regular of degree 1. It is also undirected since its corresponding suborbit is self –paired.

(c) The suborbital O_3 corresponding to the suborbit
Δ_3 is $O_3 = \{(g(1,2), g(3,1))|g \in G\}$.
The suborbital graph Γ_3 corresponding to the suborbital O_3 has a directed edge from the ordered pair V to the ordered pair W, if and only if the first co-ordinate of V is identical to the second co-ordinate of W and the second coordinate of V is not identical to the first co-ordinate of W.

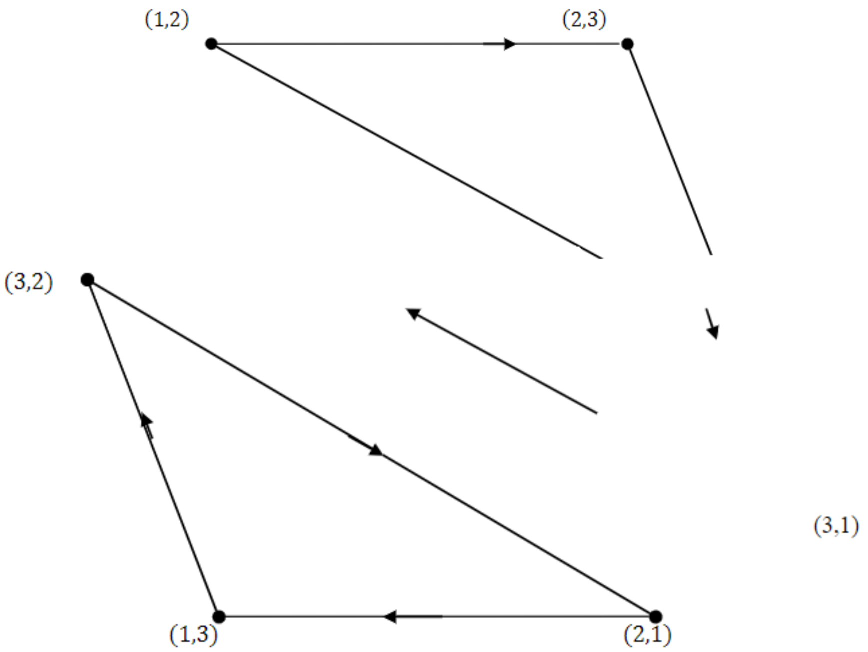
Figure 3: The suborbital graph Γ_3 corresponding to the Suborbital O_3



Γ_3 is directed and disconnected. Its girth is 3.

(d) The suborbital O_4 corresponding to the suborbit Δ_4 is $O_4=\{(g(1,2), g(2,3))|g \in G\}$.
The suborbital graph Γ_4 corresponding to the suborbital O_4 has a directed edge from the ordered pair V to the ordered pair W if and only if the second co-ordinate of V is identical to the first co-ordinate of W and the first co-ordinate of V is not identical to the second co-ordinate of W.

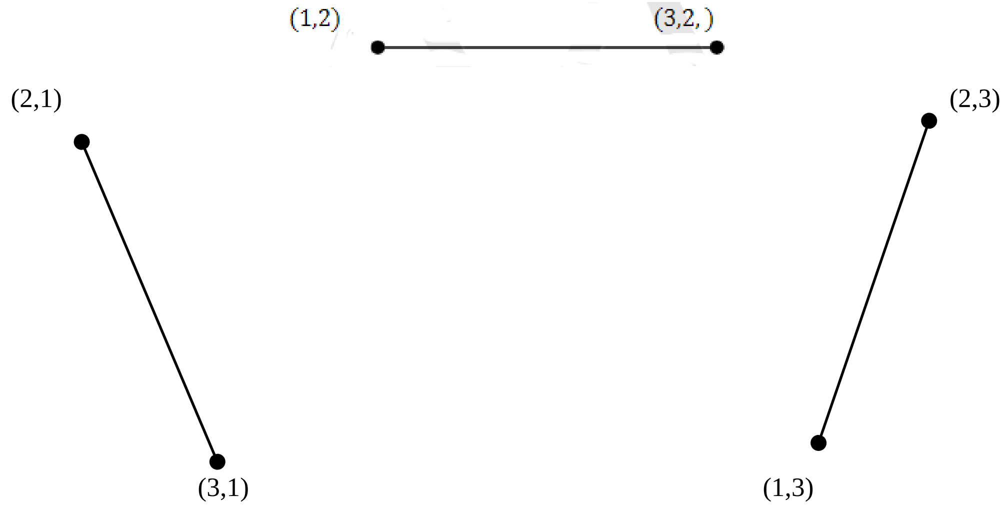
Figure 4: The suborbital graph Γ_4 corresponding to the suborbital O_4



Γ_4 is directed and disconnected. Its girth is 3.
Note that O_3 and O_4 are paired with each other.

(e) The suborbital O_5 corresponding to the suborbit Δ_5 is $O_5 = \{(g(1,2), g(2,3)|g\in G\}$.
The suborbital graph Γ_5 corresponding to the suborbital O_5 has a directed edge from the ordered pair V to ordered pair W if and only if the second co-ordinate of V is identical to the second co-ordinate of W and the first co-ordinate of V is not identical to the first co-ordinate of W.

Figure 5: The suborbital graph Γ_5 corresponding to the suborbital O_5



Γ_5 is disconnected , regular of degree 1. Γ_5 is also undirected since its corresponding suborbit is self-paired.

Since all the suborbital graphs are disconnected, G acts imprimitively on the set of ordered pairs from X = {1,2,3}.

2.2.2 SUBORBITS OF G = S_4 ACTING ON $X^{[2]}$ AND THE CORRESPONDING SUBORBITAL GRAPHS

Lemma 2.2.2.1

G acts transitively on $X^{[2]}$.

Proof

Let g ∈ G have cycle type ($\propto_1,\propto_2, \propto_3, \propto_4$), then the number of permutations in G having the same cycle type as g is given by Theorem 1.3.10.

The number of elements in $X^{[2]}$ fixed by g is given by Theorem 2.1.1.

We now have the following table;

Table 1: Permutations in G and the number of points fixed by g ∈ G

Permutation g ∈G	No. of permutations	$\lvert Fix(g)\rvert$ in $X^{[2]}$	Cycle type $(\propto_1,\propto_2,\propto_3,\propto_4)$
1	1	6	(4,0,0,0)
(ab)	6	2	(2,1,0,0)
(abc)	8	0	(1,0,1,0)
(abcd)	6	0	(0,0,0,1)
(ab)(cd)	3	2	(0,2,0,0)

Now applying Cauchy-Frobenius Lemma we have the number of orbits of G acting on $X^{[2]}$

$$= \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$$

$$= \frac{1}{24}(\,(1x12) + (6x2) + (8x0) + (6x0) + (3x0))$$

$$= \frac{1}{24}(12+12) = \frac{1}{24} \text{ x } 24 = 1\,.$$

Therefore G acts transitively on $X^{[2]}$.

□

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.3.9) to prove the Lemma above. In this case we need to show that the length of the orbit of a point say (1,2) ∈ $X^{[2]}$ is 12, the same as the number of points in $X^{[2]}$, meaning that the action of G on $X^{[2]}$ has only one orbit.

Let g∈ G have cycle type ($\propto_1, \propto_2,\propto_3, \propto_4$), then the number of permutations in G fixing (1,2) and having the same cycle type as g is given by Theorem 2.1.2

We now have the following table;

Table 2: Number of permutations in G fixing (1,2)

Permutation	$\lvert Fix(g)\rvert$ in $X^{[2]}$	Cycle type ($\propto_1, \propto_2, \propto_3, \propto_4$)
1	1	(4,0,0,0)
(ab)	1	(2,1,0,0)
(abc)	0	(1,0,1,0)
(abcd)	0	(0,0,0,1)
(ab)(cd)	0	(0,2,0,0)
Total	2	

Therefore $\lvert stab_G(1,2)\rvert = 2$

Now, applying the Orbit-Stabilizer Theorem, we get

$$\lvert Orb_G\{1,2\}\rvert = \lvert G : stab_G(1,2)\rvert = \frac{\lvert G\rvert}{\lvert stab_G(1,2)\rvert} = \frac{24}{2} = 12.$$

Thus the orbit of (1,2) is the whole of $X^{[2]}$. □

Lemma 2.2.2.2

The number of orbits of $G_{(1,2)}$ acting on $X^{[2]}$ is 7.

Proof

The Cauchy – Frobenius Lemma helps in counting the number of orbits.

The second and the third columns of the following table can be got by applying Theorems 2.1.2 and 3.1.1 respectively.

Table 3 Permutations in $G_{(1,2)}$ and the number of fixed points

Permutations g in $G_{(1,2)}$	No. of permutation	$\lvert Fix(g)\rvert$ in $X^{[2]}$
1	1	12
(1) (2) (cd)	1	2
Total	2	

$\lvert G_{(1,2)}\rvert = 2$.

By the Cauchy − Frobenius Lemma,we have

$$\left\lvert Orb_{G_{(1,2)}}(1,2)\right\rvert = \frac{1}{\lvert G_{(1,2)}\rvert}\sum_{g\in G}\lvert Fix(g)\rvert = \frac{1}{2}[(1x12) + (1x2)]=7.$$

Therefore the rank of G acting on $X^{[2]}$ is 7. □

These seven orbits are;

$Orb_{G(1,2)}(1,2) = \{(1,2)\} = \Delta_0$, the trivial orbit.

$Orb_{G(1,2)}(2,1) = \{(2,1)\} = \Delta_1$, the transpose of the trivial orbit.

$Orb_{G(1,2)}(1,3) = \{(1,3), (1,4)\} = \Delta_2$, the set of all ordered pairs containing exactly one 1 and of the form (1,a), $a \neq 2$.

$Orb_{G(1,2)}(3,1) = \{(3,1), (4,1)\} = \Delta_3$, the set of all ordered pairs containing exactly one 1 and of the (a,1), $a \neq 2$, the transpose of (1,a).

$Orb_{G(1,2)}(2,3) = \{(2,3), (2,4)\} = \Delta_4$, the set of all ordered pairs containing exactly one 2 and of the form (2,b), $b \neq 1$.

$Orb_{G(1,2)}(3,2) = \{(3,2), (4,2)\} = \Delta_5$, the set of all ordered pairs containing exactly one 2 and of the form (b,2), $b \neq 1$, the transpose of (2,b).

$Orb_{G(1,2)}(3,4) = \{(3,4), (4,3)\} = \Delta_6$, the set of all ordered pairs containing neither 1 nor 2.

Therefore the subdegrees of G acting on $X^{[2]}$ are 1, 1, 2, 2, 2, 2, and 2.

We now discuss the suborbital graphs corresponding to the suborbits determined above.

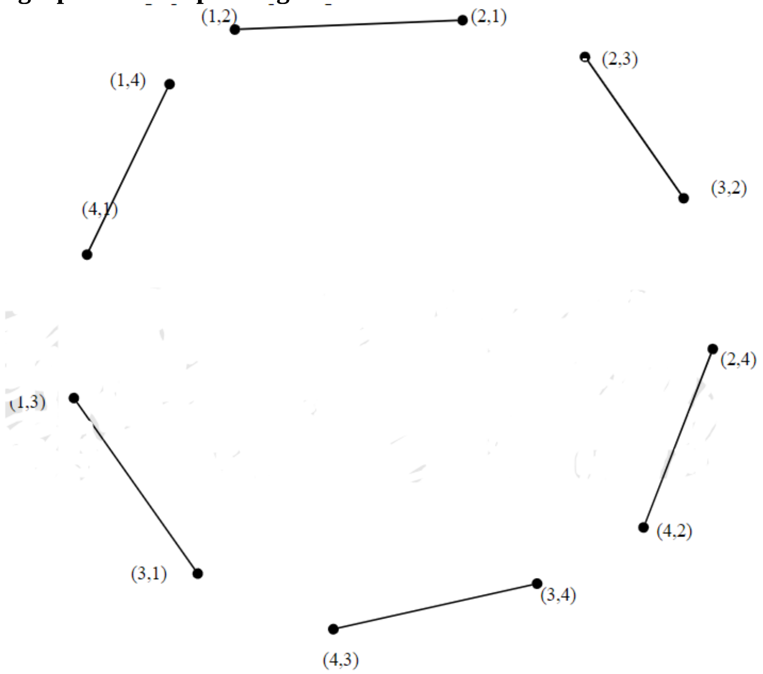
The suborbital graph corresponding to Δ_o is the null graph.

We now, construct the suborbital graphs corresponding to the suborbits Δ_1, Δ_2, Δ_3, Δ_4, Δ_5 and Δ_6 as follows; Let V and W be any two distinct ordered pairs from X = {1,2,3,4}. Then

(a) The suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{(g(1,2),g(2,1)) \mid g \in G\}$ (see Section 1.5).

Therefore in Γ_1, the suborbital graph corresponding to O_1, there is a directed edge form V to W if and only if the first co-ordinate of V is identical to the second co-ordinate of W and the second co-ordinate of V is identical to the first co-ordinate of W.

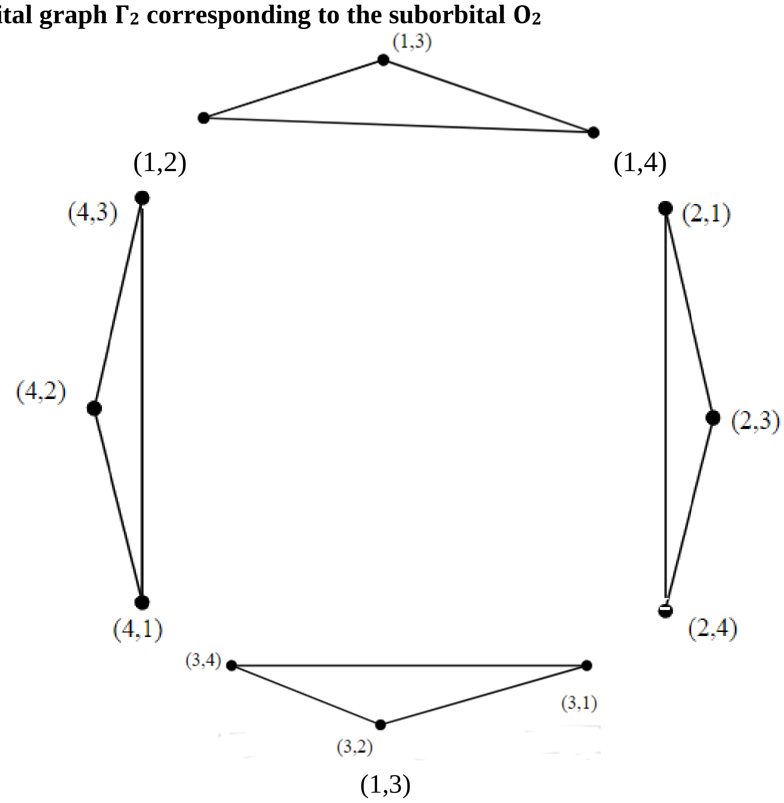
Figure 1: The suborbital graph Γ_1 corresponding to the suborbital O_1



Γ_1 is disconnected and regular of degree 1. Γ_1 is also undirected since its corresponding suborbit is self-paired.

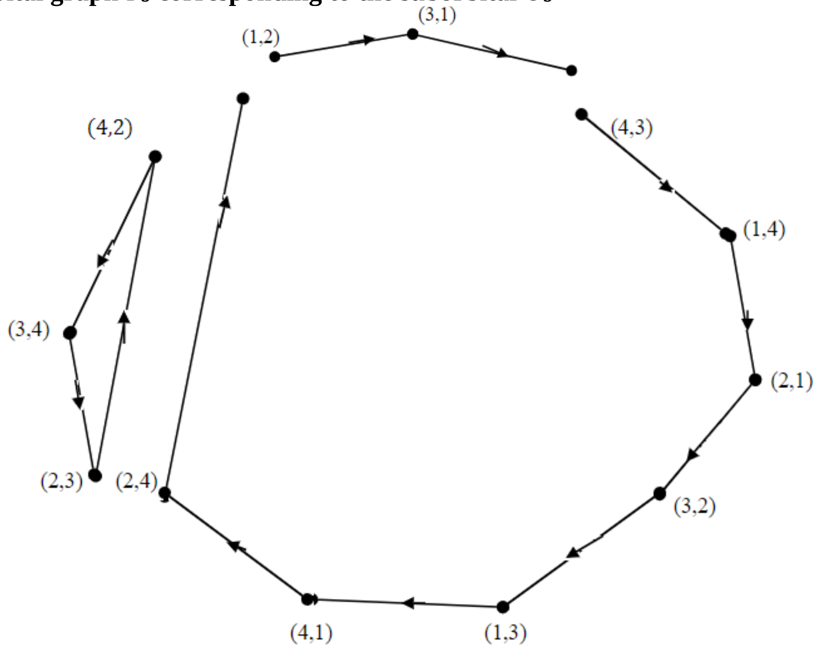
(b)The suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g\ (1,2),g(1,3))\ |g \in G\}$.
The suborbital graph Γ_2 corresponding to the suborbital O_2 has a directed edge from the ordered pair V to the ordered pair W if and only if the first coordinate of V is identical to the first co-ordinate of W and the second co-ordinate of V is not identical to the second co-ordinate of W.

Figure 2: The suborbital graph Γ_2 corresponding to the suborbital O_2



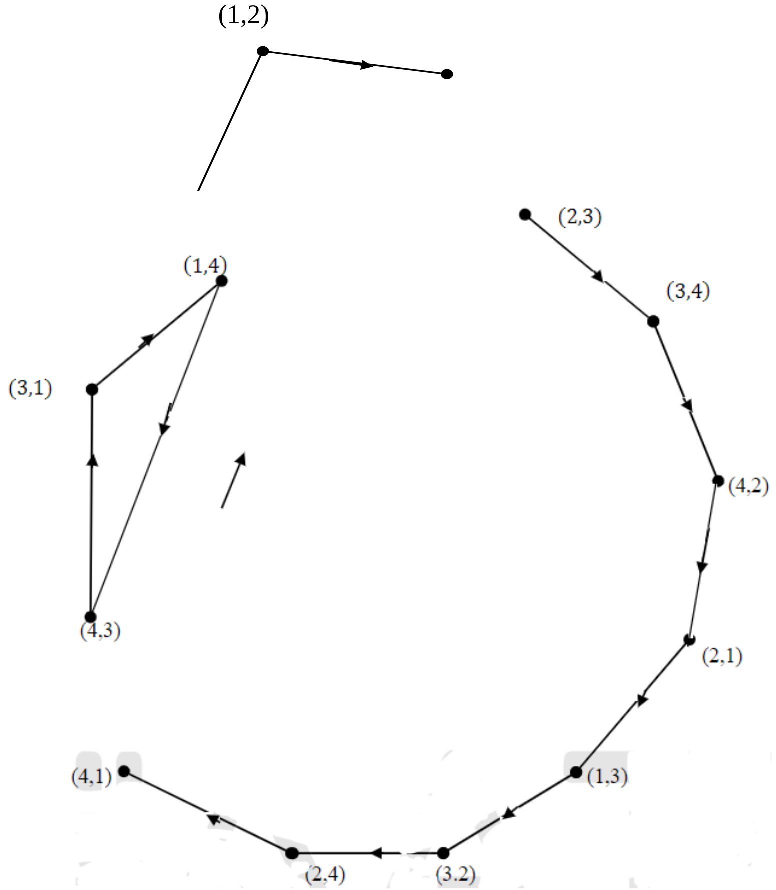
Γ_2 is disconnected and regular of degree 2. Γ_2 is also undirected since its corresponding suborbit is self-paired. Its girth is 3.
(c) The suborbital O_3 corresponding to the suborbit Δ_3 is $O_3 = \{(g\ (1,2),g(1,3))|g \in G\}$.
Therefore, the suborbital graph Γ_3 corresponding to O_3 has a directed edge from V to W if and only if the first co-ordinate of V is identical to the second co-ordinate of W and the second co-ordinate of V is not identical to the first co-ordinate of W.

Figure 3: The suborbital graph Γ_3 corresponding to the suborbital O_3



Γ_3 is directed, it is disconnected.
Its girth is 3.
(d) The suborbital O_4 corresponding to the suborbit Δ_4 is $O_4 = \{(g\ (1,2),g(1,3))|g \in G\}$.
The suborbital graph Γ_4 corresponding to the suborbital O_4 has a directed edge from the ordered pair V to the ordered pair W if and only if the second co-ordinate of V is identical to the first co-ordinate of W and the first co-ordinate of V is not identical to the second co-ordinate of W.

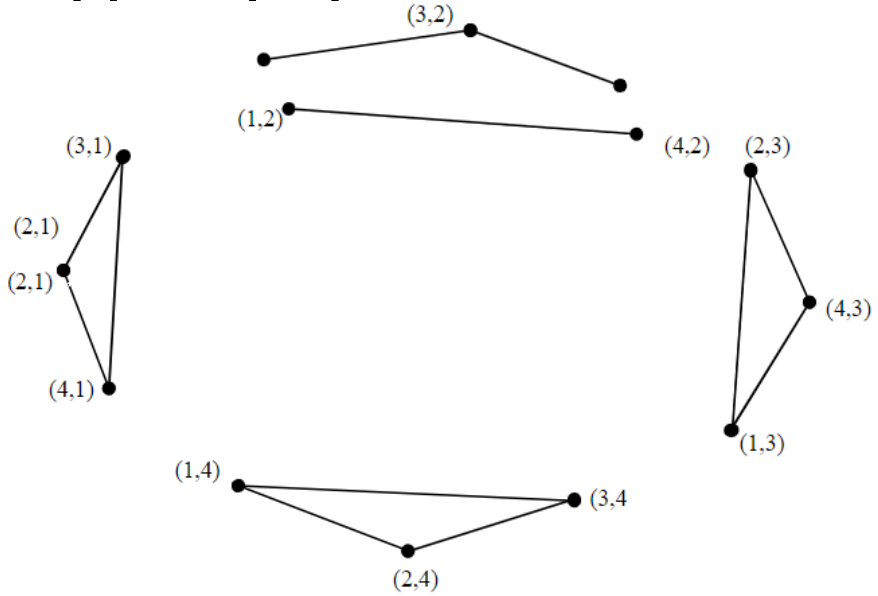
Figure 4: The suborbital graph Γ_4 corresponding to the suborbital O_4



Γ_4 is directed, it is disconnected.
Its girth is 3.
Note that O_3 and O_4 are paired with each other.
(e) The suborbital O_5 corresponding to the suborbit Δ_5 is $O_5 = \{(g(1,2), g(3,2))|g \in G\}$.

The suborbital graph Γ_5 corresponding to the suborbital O_5 has a directed edge from the ordered pair V to the ordered pair W if and only if the second co-ordinate of V is identical to the second co-ordinate of W and first co-ordinate of V is not identical to the first co-ordinate of W.

Figure 5 The suborbital graph Γ_5 corresponding to the suborbital O_5



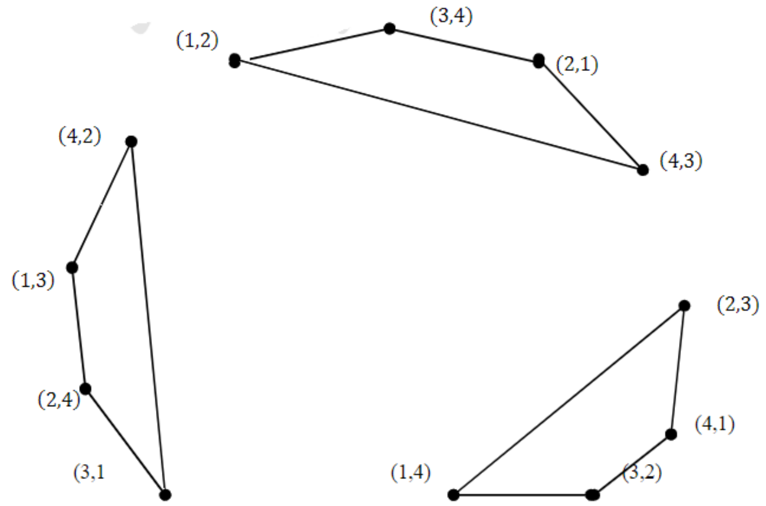
Γ_5 is disconnected, regular of degree 2. Γ_5 is also undirected since its corresponding suborbit is self-paired. Its girth is 3.

(f) And finally, the suborbital O_6 corresponding to the suborbit Δ_6 is

$$O_6 = \{(g(1,2), g(3,4))|g \in G\}.$$

The suborbital graph corresponding to the suborbital O_6 has a directed edge from the ordered pair V to the ordered pair W if and only if the co-ordinates of V are not identical to the co-ordinates of W.

Figure 6: The suborbital graph corresponding to the suborbital O_6



Γ_6 is disconnected, regular of degree 2. Γ_6 is also undirected since its corresponding suborbit is self-paired. Its girth is 4.

Since the suborbital graphs Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 and Γ_6 are all disconnected, G acts imprimitively on the set of all ordered pairs from X = {1,2,3,4}.

2.2.3 SUBORBITS OF G = S_5 ACTING ON $X^{[2]}$ AND THE CORRESPONDING SUBORBITAL GRAPHS Lemma 2.2.3.1

G acts transitively on $X^{[2]}$.

Proof

Let g∈ G have cycle type ($\propto_1$, $\propto_2$, $\propto_3$, $\propto_4$, $\propto_5$). Then the number of permutations in

G having the same cycle type as g is given by Theorem 1.3.10

The number of elements in $X^{[2]}$ fixed by g is given by Theorem 2.1.1.

We now have the following table;

Table 1: Permutations in G and the number of points fixed by g ∈ G

Permutation $g \in G$	No. of permutations	$\lvert Fix(g) \rvert$ in $X^{[2]}$	Cycle type $(\propto_1, \propto_2, \propto_3, \propto_4, \propto_5)$.
1	1	26	(5,0,0,0,0)
(ab)	10	6	(3,1,0,0,0)
(abc)	20	2	(2,0,1,0,0)
(abcd)	30	0	(1,0,0,1,0)
(abcde)	24	0	(0,0,0,0,1)
(ab) (cd)	15	0	1,2,0,0,0)
(ab) (cde)	20	0	0,1,1,0,1)
Total	120		

Now, applying Cauchy- Frobenius Lemma we obtain; number of orbits of G acting on $X^{[2]}$

$$= \frac{1}{|G|} \sum_{g \in G} |Fix\ (g)|$$

$$= \frac{1}{120} [(1x20) + (10x6) + (20x2)]$$

$$= \frac{1}{120} (20 + 60 + 40) = \frac{1}{120} (120) = 1.$$

Therefore G acts transitively on $X^{[2]}$. □

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.3.9) to prove the lemma above. In this case we need to show that the length of the orbit of a point say (1,2) ∈ $X^{[2]}$ is 20, the same as the number of points in $X^{[2]}$, meaning that the action of G on $X^{[2]}$ has only one orbit.

Let g∈G have cycle type $(\propto_1, \propto_2, \propto_3, \propto_4, \propto_5)$, then the number of permutations in G fixing (1,2) and having the same cycle type as g is given by Theorem 2.1.2

We now have the following table;

Table 2: Number of permutations in G fixing (1,2)

Permutation type	No. fixing (1,2)	Cycle type $(\propto_1, \propto_2, \propto_3, \propto_4, \propto_5)$
1	1	(5,0,0,0,0)
(ab)	3	(3,1,0,0,0)
(abc)	2	(2,0,1,0,0)
(abcd)	0	(1,0,0,1,0)
(abcde)	0	(0,0,0,0,1)
(ab) (cd)	0	(1,2,0,0,0)
(ab) (cde)	0	(0,1,1,0,1)
Total	6	

Therefore $|stab_G(1,2)| = 6$.

Now applying the Orbit-Stabilizer Theorem we get

$$|Orb_G(1,2)| = |G: stab_G(1,2)|$$

$$= \frac{|G|}{|stab_G(1,2)|} = \frac{120}{6} = 20.$$

Thus the orbit of (1,2) is the whole of $X^{[2]}$. Therefore G acts transitively on $X^{[2]}$. □

Lemma 2.2.3.2

The number of orbits of $G_{(1,2)}$ acting on $X^{[2]}$ is 7.

Proof

The Cauchy-Frobenius Lemma helps in counting the number of orbits.

The second and the third columns of the following table can be got by applying Theorems 2.1.2 and 2.1.1 respectively.

Table 3: Permutations in $G_{(1,2)}$ and the number of fixed points

Permutation g in $G_{(1,2)}$	No. of permutations	Fix (g) in $X^{[2]}$
1	1	20
(1) (2) (cde)	2	2
(1) (2) (cd) (e)	3	6
Total	6	

By Cauchy-Frobenius Lemma, we have

$$|OrbG(1,2)(1,2)| = |G|(1G,|2)| \sum |Fix\ (g)|$$

$$= \frac{1}{6} (1X20) + (2X2) + (3X6)$$

$$= \frac{1}{6}(20+4+18) = \frac{1}{6} \times 42 = 7.$$

Therefore the rank of G acting of $X^{[2]}$ is 7. □

These seven orbits are;

$Orb_{G(1,2)}(1,2) = \{(1,2)\} = \Delta_0$, the trivial orbit.

$Orb_{G(1,2)}(2,1)$ = {(2,1) }= Δ_1, the transpose of the trivial orbit.
$Orb_{G(1,2)}(1,3)$ = {(1,3), (1,4), (1,5) } = Δ_2, the set of all ordered pairs containing exactly one 1 and of the form (1,a), $a \neq 2$.
$Orb_{G(1,2)}(3,1)$ = {(3,1), (4,1), (5,1) } = Δ_3, the set of all ordered pairs containing exactly one 1 and of the form (a,1), $a\neq 2$, the transpose of (1,a).
$Orb_{G(1,2)}(2,3)$ = {(2,3), (2,4), (2,5) } = Δ_4, the set of all ordered pairs containing exactly one 2 and of the form (2,b), $b\neq 1$.
$Orb_{G(1,2)}(3,2)$ = {(3,2), (4,2), (5,2,) } = Δ_5, the set of all ordered pairs containing exactly one 2 and of the form (b,2), $b\neq 1$, the transpose of (2,b)
$Orb_{G(1,2)}(3,4)$ = {(3,4), (3,5), (4,3), (4,5), (5,4) } = Δ_6, the set of all ordered pair containing neither 1 nor 2. Thus the subdegrees of G acting on $X^{[2]}$ are 1, 1, 3, 3, 3, 3, and 6.

Now, we discuss the suborbital graphs corresponding to the suborbits determined above. The suborbital graph corresponding to Δ_0 is the null graph. We are now left with the suborbits Δ_1, Δ_2, Δ_3, Δ_4, Δ_5 and Δ_6.
Since $|G_{(1,2)}| = 6$ is even, by Theorem 1.5.3, $G_{(1,2)}$ has at least an orbit different from Δ_0 which is paired with itself. In fact Δ_1, Δ_2, Δ_5, and Δ_6 are self-paired.
By the theory developed in Section 1.5, the suborbital graphs corresponding to these suborbits are undirected.
Δ_3 and Δ_4 are not self-paired. In fact they are paired with each other.
The suborbital graphs corresponding to the suborbits Δ_3 and Δ_4 are directed.
We now describe how to construct these suborbital graphs.
Let V and W be any two distinct ordered pairs from X = {1,2,3,4,5}. Then

(a) The suborbital O_1 corresponding to the suborbit Δ_1 is
$O_1 = \{(g(1,2), g(2,1))|g \in G\}$ (See section 1.5).
In Γ_1 the suborbital graph corresponding to O_1, there is a directed edge from ordered pair V to ordered pair W if and only if the first co-ordinate of V is identical to the second co-ordinate of W and the second co-ordinate of V is identical to the first co-ordinate of W.
Γ_1 has no cycles. It is disconnected and regular of degree 1.

(b) The suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g(1,2), g(1,3))|g \in G\}$
In Γ_2 the suborbital graph corresponding to O_2 has a directed edge from the ordered pair V to the ordered pair W if and only if the first co-ordinate of V is identical to the first co-ordinate of W and the second co-ordinate of V is not identical to the second co-ordinate of W.
The graph Γ_2 is undirected since O_2 is self-paired. It is disconnected, regular of degree 2. Its girth is 3.

(c) The suborbital O_3 corresponding to the suborbit Δ_3 is $O_3 = \{(g(1,2),g(3,1))|g \in G\}$.
Therefore, the suborbital graph Γ_3 corresponding to O_3 has a directed edge from the ordered pair V to the ordered pair W if and only if the first co-ordinate of V is identical to the second co-ordinate of W and the second co-ordinate of V is not identical to the first co-ordinate of W. The suborbital graph Γ_3 is directed.

(d)The suborbital O_4 corresponding to the suborbit Δ_4 is $O_4 = \{(g(1,2), g(2,3))|g \in G\}$.
The suborbital graph Γ_4 corresponding to the suborbital O_4 has a directed edge from the ordered pair V to the ordered pair W if and only if the second co-ordinate of V is identical to the first co-ordinate of W and the first co-ordinate of V is not identical to the second co-ordinate of W. The suborbital graph Γ_4 is directed.

(e) The suborbital O_5 corresponding to the suborbit Δ_5 is $O_5 = \{(g(1,2), g(3,2))|g \in G\}$.
The suborbital graph Γ_5 corresponding to the suborbital O_5 has a directed edge from the ordered pair V to the ordered pair W if and only if the second co-ordinate of V is identical to the second co-ordinate of W and the first co-ordinate of V is not identical to the first co-ordinate of W.
Γ_5 is undirected. It is disconnected. It is regular of degree 2. Its girth is 3.

(f) Finally, the suborbital O_6 corresponding to the suborbit Δ_6 is $O_6 = \{(g(1,2), g(3,4))|g \in G\}$.
The suborbital graph Γ_6 corresponding to the suborbital O_6 has a directed edge from the ordered pair V to the ordered pair W if and only if the co-ordinates of V are not identical to the co-ordinates of W. Γ_6 is undirected, it is disconnected and regular of degree 2. Its girth is 4.
The suborbital graphs Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 and Γ_6 are all disconnected. Therefore G acts imprimitively on the set of all ordered pairs from X = {1,2,3,4,5}.

CONCLUSIONS

In this research paper, we have discussed some properties of S_n, (n = 3, 4, 5) acting on ordered pairs.
We found out that S_n (n = 3, 4, 5) acts on $X^{[2]}$ transitively but not primitively.
The rank of S_3 is 6 and has subdegrees 1,1,1,1,1,1. The suborbital graphs corresponding to the five non-trivial suborbits are: Γ_1, Γ_2, Γ_5, which are undirected and Γ_3 and Γ_4 which are directed.
S_4 and S_5 each has a rank of 7. Their subdegrees are 1,1,2,2,2,2,2 and 1,1,3,3,3,3,6 respectively.

The suborbital graphs of the non-trivial suborbits for S_4 and S_5 are all disconnected.
Their suborbital graphs Γ_1, Γ_2, Γ_5, Γ_6 are all undirected since their corresponding suborbits are self-paired while Γ_3 and Γ_4 are directed. In fact Γ_3 and Γ_4 are paired with each other.
From this work and what was done by others on S_6 and S_7, we can conjecture that S_n $(n \geq 4)$ acts on $X^{[2]}$ transitively and the rank is 7.Finally we also conjecture that the subdegrees, when S_n $(n \geq 4)$ acts on $X^{[2]}$ are:

$$1, 1, n\text{-}2, n\text{-}2, n\text{-}2, n\text{-}2 \text{ and } 2\binom{n}{2} - [4(n-2)+2].$$

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