FUZZY TOPOLOGICAL MODULES INDUCED BY FUZZY PSEUDO NORMED MODULE

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Abstract
In this article, we give defines about fuzzy pseudo norm module space and fuzzy metric module space which induces by this space. Complete metric space was proved. Also we are defining fuzzy topological module space which induces by fuzzy metric module space.

Keyword: Fuzzy pseudo norm module space, fuzzy metric module space, fuzzy scalar, complete fuzzy metric module space, fuzzy topological module space
INTRODUCTION:


In the present work we defining fuzzy pseudo norm module space and defining fuzzy metric module space which induces by fuzzy norm module space and using the concept of fuzzy scalar to measure the distances between fuzzy points. Also we are giving fuzzy topological module space which induces by fuzzy metric module space.

To rich the article some fundamental of fuzzy set and point are gave below. The symbol $J$ will denote the closed interval $[0,1]$. Let $M$ be a non-empty set.

Definition [10] 1.1

The term fuzzy set in a set $M$ is a map $A: M \rightarrow J$, that is, an element of $J^M$. Let $E \in J^M$. $m \in M$, we symboled by $E(m)$ or $m_a$ of the membership degree of $m$ in $E$. If $E(m)$ belong to $\{0,1\}$, then $E$ is a crisp set in $M$.

Definition [8] 1.2

A pair $(M, \tau_M)$, where $M$ is R- module and $\tau_M$ a topology on $M$, is a called a topological module if the following maps are continuous:
1) $R \times R \rightarrow R, (r,k) \rightarrow r + k$.
2) $R \rightarrow R, r \rightarrow -r$
3) $R \times M \rightarrow M, (m,k) \rightarrow m.k$

Definition [1] 1.3

A class $\tau_F \in J^M$ of fuzzy set is called a fuzzy topology for $M$ if the following are hold:
1) $\emptyset, M \in \tau_F$
2) $\forall A, B \in \tau_F \rightarrow A \cap B \in \tau_F$
3) $\forall (A_j)_{j \in J} \in \mu \rightarrow \vee_{j \in J} A_j \in \tau_F$

$(M, \tau_F)$ is called fuzzy topological space. The set $E$ is fuzzy open if $E \in \tau_F$, and complement of $E$ is an fuzzy closed.

Definition 1.4 [7]

Let $R$ be a ring and let $M$ be a left $R$-module. A fuzzy set $E$ in $M$ is called a fuzzy left $R$-module if for each $m, n \in M$ and $r \in R$:
(1) $E(m + n) \geq \min\{E(m), E(n)\}$.
(2) $E(m) = E(m^{-1})$.
(3) $E(rm) \geq E(m)$.
(4) $E(0) = 1$.

Definition 1.5 [7]

Let $R$ be a fuzzy topological ring, the set $M$ is said to be left fuzzy topological module on the fuzzy topological ring $R$ if:
(1) $E$ left fuzzy module on $R$.
(2) $E$ is a fuzzy topology compatible with the stricture of fuzzy group on $E$ and satisfies the following axiom:

The mapping $R \times M \rightarrow M$ defined by $(\lambda, m) \rightarrow \lambda m$ is a fuzzy continuous.

Definition 1.6 [9]

Let $P_F(M)$ be the set of all the fuzzy point $m_a$ of $M$. Especially, if $M = \mathbb{R}$ (the real space), we say that fuzzy scalars instead of fuzzy points. We symboled by $S_F(\mathbb{R})$ for all fuzzy scalars set and we symboled by $S_F^+(\mathbb{R})$ for all positive fuzzy scalars set.

Definition 1.7 [9]

Suppose that $M$ is a fuzzy linear space. $(M, \pi_F)$ is said to be a fuzzy normed linear space if the following mapping $\pi_F: P_F(M) \rightarrow S_F^+(\mathbb{R})$ satisfies:
1) $\pi_F(m_a) = 0$ if $m = 0$ and $a = 1$
2) For any $r \in \mathbb{R}$ and $m_a \in M$ then $\pi_F(rm_a) = |r| \pi_F(m_a)$
3) For any $m_a, n_a \in M$, $\pi_F(m_a, n_a) \leq \pi_F(m_a) + \pi_F(n_a)$
Definition 1.8 [9]  
Suppose that \((M, \pi_F)\) is a fuzzy linear space defined on \(M\). The fuzzy norm \(\pi_F\) of any fuzzy point \(m_a\) in \(M\) is defined by:
\[\pi_F(m_a) = (\pi(m), \alpha), \quad \forall m_a \in M\]

Where \(\pi(m)\) is the norm of \(m\) defined in \((M, \pi)\).

Definition 1.9 [9]  
Suppose \(M\) is a nonempty set and \(d_F: P_F(M) \times P_F(M) \to S^+_k(\mathbb{R})\) is a mapping. \((P_F(M), d_F)\) is said to be a fuzzy metric space if for any \([m_a, n_\beta, h_\gamma] \in P_F(R)\), \(d_F\) satisfies the following three conditions,
1) \(d_F(m_a, n_\beta) \geq 0 \) and \(d_F(m_a, n_\beta) = 0 \) iff \(m = n\) and \(\alpha = \beta = 1\)
2) \(d_F(m_a, n_\beta) = d_F(n_\beta, m_a)\)
3) \(d_F(m_a, n_\beta) \leq d_F(m_a, h_\gamma) + d_F(h_\gamma, n_\beta)\)

Definition 1.10 [9]  
Suppose \((M, d)\) is an ordinary metric space. The distance of any two fuzzy points \(m_a, n_\beta\) in \(d_F(R)\) is defined by
\[d_F(m_a, n_\beta) = (d(m, n), \min \{\alpha, \beta\})\]

Where \((d(m, n), \min \{\alpha, \beta\})\) is the distance between \(m\) and \(n\) defined in \((M, d)\).

**Fuzzy Normed Module Space**

Definition 2.1  
Suppose that \(M\) be an R- module. \((M, \pi_{FM})\) is said to be a fuzzy pseudo normed R-module space if the following mapping \(\pi_{FM}: P_F(M) \to S^+_k(\mathbb{R})\) satisfies:
1) \(\pi_{FM}(m_a) = 0\) iff \(r = 0\) and \(\alpha = 1\)
2) For any \(m_a, k_\beta \in M\), \(\pi_{FM}(m_a - k_\beta) \leq \pi_{FM}(m_a) + \pi_{FM}(k_\beta)\)
3) For any \(m_a, k_\beta \in M\), \(\pi_{FM}(m_a, k_\beta) \leq \pi_{FM}(m_a) \cdot \pi_{FM}(k_\beta)\)
4) For each \(r_\beta \in R\) and \(m_a \in M\) then \(\pi_{FM}(r_\beta m_a) = |r_\beta| \pi_{FM}(m_a)\)

Definition 2.2  
Suppose that \((M, \pi_M)\) is a pseudo normed R-module space defined on \(M\) then \((M, \pi_{FM})\) is a fuzzy normed module space with \(\pi_{FM}\) is a mapping from \(M\) to \(S^+_k(\mathbb{R})\) defined by:
\[\pi_{FM}(m_a) = (\pi_M(m), \alpha), \quad \forall m_a \in M\]

Where \(\pi_M(m)\) is the norm of \(m\) defined in \((M, \pi_M)\).

**Example 2.3**  
Let \((\mathbb{Z}, +, \cdot)\) be a commutative ring and \((R^n, \pi_\beta)\) be a linear normed \(\mathbb{Z}\) -module space defined in \(R^n\) (n-dim. Euclidean space). The fuzzy norm of \(m_a \in R^n\), is defined by \(\pi_{FM}(m_a) = h_\alpha\) where \(h_\alpha \in S^+_k(\mathbb{R})\) such that \(h = \pi_M(r)\). Then \((P_F(R^n), \pi_{FM}(m_a))\) is a fuzzy normed module space.

**Theorem 2.4**  
Suppose \((P_F(M), \pi_{FM}(m))\) be a fuzzy normed R-module space, then the function \(d_{FM}(m_a, k_\beta) = \pi_{FM}(m_a - k_\beta) = (\pi_M(m - k), \min \{\alpha, \beta\})\) satisfies the fuzzy metric axioms, thus \((P_F(M), d_{FM})\) is a fuzzy metric R-module space.

**Proof**
1) \(d_{FM}(m_a, k_\beta) = 0 \iff \pi_{FM}(m_a - k_\beta) = 0 \iff m_a = k_\beta \iff m = k\) and \(\alpha = \beta = 1\)
2) \(d_{FM}(m_a, k_\beta) = \pi_{FM}(m_a - k_\beta) = \pi_{FM}(k_\beta - m_a) = d_{FM}(k_\beta, m_a)\)
3) \(d_{FM}(m_a, k_\beta) = \pi_{FM}(m_a, k_\beta) \leq \pi_{FM}(m_a) \cdot \pi_{FM}(k_\beta) \leq d_{FM}(m_a, d_{FM}(k_\beta)\)
4) \(d_{FM}(m_a, k_\beta) = \pi_{FM}(m_a - k_\beta) = \pi_{FM}(m_a - h_\gamma + h_\gamma - k_\beta) = \pi_{FM}(m_a - h_\gamma - (k_\beta - h_\gamma)) \leq \pi_{FM}(m_a - h_\gamma) + \pi_{FM}(h_\gamma - k_\beta) = d_{FM}(m_a, h_\gamma) + d_{FM}(h_\gamma, k_\beta)\)

**Example 2.5**  
Let \((\mathbb{Z}, +, \cdot)\) be a commutative ring and \((R^n, +, \cdot, d_\beta)\) be a linear normed \(\mathbb{Z}\) -module space defined in \(R^n\) (n-dim. Euclidean space). The distance between two fuzzy point \(m_a, k_\beta \in\)
Definition 2.6
Suppose \((P_F(M), d_{FM})\) is the fuzzy metric R-module space. We define a fuzzy open ball \(B_F(m_a, r_p)\) where \(m_a\) is the center of \(B_F\) with membership \(\alpha\) and radius \(r_p \in S^\alpha_F(\mathbb{R})\) as:

\[
B_F(m_a, r_p) = \{ k_{\beta} \in P_F(R) : d_{FR}(m_a, k_{\beta}) \leq r_p \}
\]

Definition 2.7
Suppose \((P_F(M), d_{FM})\) is the fuzzy metric R-module space. A fuzzy set \(E\) is said to be fuzzy open set if for each \(m_a\) s.t \(\alpha \leq E(m)\) there exist a fuzzy open ball \(B_F(m_a, r_p), \alpha \leq B(m), r_p \in S^\alpha_F(\mathbb{R})\) s.t \(B_F(m_a, r_p) \leq E(m)\)

Theorem 2.8
Suppose \((P_F(R), d_{FR})\) is the fuzzy metric R-module space. then \((P_F(M), \tau_{FM})\) is the fuzzy topological R-module space induced by \((P_F(M), d_{FM})\) where \(\tau_{FM}\) is defined by

\[
\tau_{FM} = \{ E \leq P_F(M) : E\text{ is fuzzy open set in } (P_F(M), d_{FM}) \}
\]

Proof
1) Let \(r_o \in \emptyset\) and \(k_{\alpha} \in S^\alpha_F(\mathbb{R})\) define \(B_F = \{ h_o : d(r_o, h_o) < k_{\alpha} \}\) is a fuzzy open ball, implies \(r_o \in B_F \subseteq \emptyset\). Thus \(\emptyset \in \mu\)

2) Let \(A, C \in \tau_{FM}\) and \(r_{\alpha} \in A \cap C\), then \(\alpha = \min(A(r), C(r))\) implies \(\alpha \leq A(r)\) and \(\alpha \leq C(r)\). Put \(B(r) = \min\{A(r), C(r)\}\) then there exists fuzzy open ball \(B_F(r_{\alpha}, k_{\beta})\) where \(k_{\beta} \in S^\alpha_F(\mathbb{R})\) implies \(r_{\alpha} \leq B_F(r_{\alpha}, k_{\beta}) \leq A \cap C \subseteq \tau_{FM}\)

3) Let \(A_j \in \mu, j \in J\) and \(r_{\alpha} \in \bigcup_{j \in J} A_j\), then \(\alpha \leq \sup\{A_j(r), j \in J\}\) implies that there exists \(j \in J\) s.t \(\alpha \leq A_j(r)\).

Since \(A_j \in \tau_{FM}\), then there exists fuzzy open ball \(B_F(r_{\alpha}, k_{\beta})\) where \(k_{\beta} \in S^\alpha_F(\mathbb{R})\) s.t \(\alpha \leq B(r) \leq \sup\{A_j(r), j \in J\}\). Thus \(\bigcup_{j \in J} A_j \in \tau_{FM}\)

Let’s verify that the fuzzy R-module operations are fuzzy continuous in fuzzy topological \(\tau_{FM}\) defined by this fuzzy metric \(d_{FM}\). Let \(a_o, b_o \in M\) and \(\epsilon_o \in S^\epsilon_F(\mathbb{R})\), put \(\delta_o = \min\{\frac{\epsilon_o}{2}, \frac{r_o}{2}\}\)

Let \(m_{a_o}, n_{b_o} \in M, s.t\ d_{FM}(m_{a_o} - a_o) \leq \delta_o\) and \(d_{FM}(n_{b_o} - b_o) \leq \delta_o\)

\[
d_{FM}(m_{a_o} - n_{b_o} - (a_o - b_o)) = \pi_{FM}((m_{a_o} - n_{b_o}) - (a_o - b_o))
\]

\[
\leq \pi_{FM}(m_{a_o} - n_{b_o}) + \pi_{FM}(a_o - b_o)
\]

\[
= \pi_{FM}(m_{a_o} - a_o) + \pi_{FM}(n_{b_o} - b_o) \leq \delta_o + \delta_o \leq \epsilon_o
\]

Let \(m_{a_o}, n_{b_o} \in M, s.t\ d_{FM}(m_{a_o} - a_o) \leq \delta_o\) and \(d_{FM}(n_{b_o} - b_o) \leq \delta_o\) and hence, \(\pi_{FM}(m_{a_o} - a_o) \leq \pi_{FM}(n_{b_o} - b_o) \leq \delta_o\) i.e \(\pi_{FM}(m_{a_o}) = \pi_{FM}(a_o) + \delta_o \)

\[
d_{FM}(m_{a_o} - n_{b_o} - (a_o - b_o)) = \pi_{FM}((m_{a_o} - n_{b_o}) - (a_o - b_o))
\]

\[
\leq \pi_{FM}(m_{a_o} - a_o) + \pi_{FM}(n_{b_o} - b_o) \leq \delta_o + \delta_o \leq \epsilon_o
\]

\[
d_{FM}(r_p(m_{a_o}, n_{b_o})) = \pi_{FM}(r_p(m_{a_o}, n_{b_o})) = \pi_{FM}(m_{a_o}, n_{b_o})
\]

\[
\leq \epsilon_o
\]

References
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