Using the gun of Mathematical induction to conquer some theorems of The Mulatu Numbers.

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Abstract

The Mulatu numbers are sequences of numbers of the form 4, 1, 5, 6, 11, 17, 28, 45, .... The numbers have wonderful and amazing properties and patterns. In mathematical terms, it is defined by the following recurrence relation:

\[ M_n = \begin{cases} 
4 & \text{if } n = 0; \\
1 & \text{if } n = 1; \\
M_{n-1} + M_{n-2} & \text{if } n > 1.
\end{cases} \]

The first number of the sequence is 4, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers of the sequence itself. That is, after two starting values, each number is the sum of the two preceding numbers. In this paper, we give summary of some important properties and patterns of Mulatu numbers. Its relations to the Fibonacci and Lucas numbers are also investigated.

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1. Introductions and Background

The Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, an Ethiopian Mathematician and Professor of Mathematics at Savannah State University, Savannah, Georgia, USA. The numbers are closely related to both Fibonacci and Lucas Numbers in its properties and patterns. Below we give the First 20 Mulatu, Fibonacci and Lucas numbers.
First 20 Mulatu, Fibonacci and Lucas Numbers (Tables 1 & 2).

**Table 1**

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mₙ</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td>11</td>
<td>17</td>
<td>28</td>
<td>45</td>
<td>73</td>
<td>118</td>
<td>191</td>
<td>309</td>
</tr>
<tr>
<td>Fₙ</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
</tr>
<tr>
<td>Lₙ</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>18</td>
<td>29</td>
<td>47</td>
<td>76</td>
<td>123</td>
<td>199</td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>n</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mₙ</td>
<td>500</td>
<td>809</td>
<td>1309</td>
<td>2118</td>
<td>3427</td>
<td>5545</td>
<td>8972</td>
<td>14517</td>
<td>23489</td>
</tr>
<tr>
<td>Fₙ</td>
<td>144</td>
<td>233</td>
<td>377</td>
<td>610</td>
<td>987</td>
<td>1597</td>
<td>3584</td>
<td>4181</td>
<td>6765</td>
</tr>
<tr>
<td>Lₙ</td>
<td>322</td>
<td>521</td>
<td>843</td>
<td>1364</td>
<td>2207</td>
<td>3571</td>
<td>5778</td>
<td>9349</td>
<td>15127</td>
</tr>
</tbody>
</table>

**Background materials**

In this paper M, L and F denote the Mulatu, Lucas and Fibonacci numbers respectively. We prove the theorems using induction only. The theorems are not new but the proofs are new.

The Fibonacci numbers are sequences of numbers of the form: 0, 1, 1, 2, 3, 5, 8, 13, …. In mathematical terms, it is defined by the following recurrence relation:

\[ F_n = F_{n-1} + F_{n-2} \] with \( F_1 = F_2 = 1 \) and \( F_0 = 0 \).
The first number of the sequence is 0, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers of the sequence itself.

The Lucas numbers are sequences of numbers of the form: 2, 1, 3, 4, 7, 11, 28, 29, 47, 76, 123, ....

In mathematical terms, it is defined by the following mathematical recurrence relation

\[ L_0 = 2, \]
\[ L_1 = 1, \]
\[ L_{n+1} = L_n + L_{n-1} \text{ for } n > 1. \]

Each subsequent number is equal to the sum of the previous two numbers of the sequence itself. The following identities of \( F \) and \( L \) will be used in this paper.

(1) \[ L_n = F_{n-1} + F_{n+1} \]
(2) \[ F_{n+1} = F_n + F_{n-1} \]
(3) \[ F_{2n} = F_n L_n \]
(4) \[ L_{2n} = F_n + 2F_{n-1} \]
(5) \[ F_n = \frac{L_{n+1} + L_{n-1}}{5} \]
(6) \[ L_{n+1} = L_n + L_{n-1} \]
(7) \[ F_{n+k} = F_{n-1}F_k + F_kF_{k+1} \]
(8) \[ 5F_n^2 - L_n^2 = 4(-1)^{n+1} \]
(9) \[ L_{n+m} = \frac{5F_nF_m + L_nL_m}{2} \]
(10) \[ M_{n+k} = F_{n-1}M_k + M_nF_{k+1} \]
2. The Main Results

**Theorem 1** (Partial sum of Mulatu numbers). If each \( M_i \) \((i \geq 0)\) are Mulatu numbers, then

\[
\sum_{k=0}^{n} M_k = M_{n+2} - 1.
\]

**Proof:** We use indication on \( n \)

1. When \( n = 1 \), the formula is true as \( M_0 + M_1 = 4 + 1 = 5 = M_3 - 1 \).
2. Assume the formula is true for \( n = p \).
3. Verify the formula for \( n = p + 1 \)

\[
\sum_{k=0}^{p+1} M_k = \sum_{k=0}^{p} M_k + M_{p+1} = M_{p+2} - 1 + M_{p+1} = M_{p+1} + M_{p+2} - 1 = M_{p+3} - 1.
\]

Hence the Theorem follows by induction.

**Theorem 2** (Partial sum of the Mulatu numbers with odd indices). If each \( M_i \) \((i \geq 0)\) are Mulatu numbers, then

\[
\sum_{k=0}^{n} M_{2k+1} = M_{2n+2} - 4
\]

**Proof:** We use indication on \( n \).

1. When \( n = 1 \), the formula for \( n = 1 \) is true \( M_1 + M_3 = M_4 - 4 = 7 \).
2. Assume the formula is true for \( n = p \).
3. Verify the formula for \( n=p+1 \)

\[
\sum_{k=0}^{p+1} M_{2k+1} = \sum_{k=0}^{p} M_{2k+1} + M_{2p+3} \\
= M_{2p+2} - 4 + M_{2p+3} \\
= M_{2p+4} - 4
\]

Hence, the Theorem follows by induction.

**Theorem 3** (Partial sum of the Mulatu numbers with even indices). If each \( M_i \ (i \geq 0) \) are Mulatu numbers, then

\[
\sum_{k=0}^{n} M_{2k} = M_{2n+1} + 3.
\]

**Proof:** We use indication on \( n \).

1. When \( n=1 \), the formula for \( n=1 \) is true as \( M_0 + M_2 = M_3 + 3 = 9 \)
2. Assume the formula is true for \( n=p \)
3. Verify the formula for \( n=p+1 \). Note that

\[
\sum_{k=0}^{p+1} M_{2k} = \sum_{k=0}^{p} M_{2k} + M_{2p+2} \\
= M_{2p+1} + 3 + M_{2p+2} \\
= M_{2p+3} + 3
\]

Hence by induction, the Theorem follows.

**Theorem 4** (Expressing \( M \) in terms of \( F \)). Let \( M_n \) and \( F_n \) be any Mulatu and Fibonacci numbers. Then we have

\[
M_n = F_{n-3} + F_{n-1} + F_{n+2}.
\]
**Proof.** We use induction on \( n \).

(1) When \( n = 0 \), the formula is true as 
\[
M_0 = F_{-3} + F_{-1} + F_2
\]
and using 
\[
F_{-n} = (-1)^{n+1} F_n,
\]
we have 
\[
4 = 2 + 1 + 1 = 4.
\]

(2) Assume the formula is true for \( n = 1, 2, 3, \ldots, k-1, k \).

(3) Verify the formula for \( n = k+1 \).

Note that 
\[
M_{k+1} = M_k + M_{k-1} = F_{k-3} + F_{k-1} + F_{k+2} + F_{k-4} + F_{k-2} + F_{k+1}
\]
\[
= M_k + M_{k-1} = F_{k-4} + F_{k-3} + F_{k-2} + F_{k-1} + F_{k+2} + F_{k+1}
\]
\[
= F_{k-2} + F_k + F_{k+3}
\]
Hence by induction, the theorem follows.

**Theorem 5.** 
\[
L_n = \frac{M_n + F_n}{2}
\]

**Proof:** We use induction on \( n \).

1. When \( n=1 \), the formula is true as 
\[
L_1 = \frac{M_1 + F_1}{2} = \frac{1+1}{2} = 1
\]

2. Assume the formula is true for \( n=1, 2, 3 \ldots, k-1, k \)

3. Verify the formula for \( n=k+1 \).

\[
L_{k+1} = L_k + L_{k-1}
\]
\[
= \frac{M_k + F_k}{2} + \frac{M_{k-1} + F_{k-1}}{2}
\]
\[
= \frac{M_k + M_{k-1} + F_k + F_{k-1}}{2}
\]
\[
= \frac{M_{k+1} + F_{k+1}}{2}
\]

Hence by induction, the theorem follows.
Theorem 6. \( M_n = 4F_{n+1} - 3F_n \)

**Proof:** We use indication on \( n \).

1. When \( n=1 \), the formula for \( n=1 \) is true as \( M_1 = 4F_2 - 3F_1 = 1 \)

2. Assume the formula is true for \( n=k \)

3. Verify the formula for \( n=k+1 \). Note that

\[
M_{k+1} = M_k + M_{k-1} = (4F_{k+1} - 3F_k) + (F_k - 3F_{k-1})
\]

\[
= 4(4F_{k+1} + F_k) - 3(F_k + F_{k-1})
\]

\[
= 4F_{k+2} - 3F_{k+1}
\]

Hence by induction, the Theorem follows.

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**References:**
